Comparison of interacting diffusions and an application to their ergodic theory

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Abstract

A general comparison argument for expectations of certain multi-time functionals of infinite systems of linearly interacting diffusions differing in the diffusion coefficient is derived. As an application we prove clustering occurs in the case when the symmetrized interaction kernel is recurrent, and the components take values in an interval bounded on one side. The technique also gives an alternative proof of clustering in the case of compact intervals.

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1 Introduction and results

In [FG94a] and implicitly in [CG94], a comparison of general linearly interacting diffusions with interacting Fisher-Wright diffusions was a powerful tool in the study of the long-term behavior of a class of models differing in the diffusion coefficient, in particular in establishing *universality properties.* For this it was important that one of the models in the comparison was Fisher-Wright, since a duality argument with delayed coalescing random walks was involved.

Here we provide a general method based on the intuition that a larger diffusion coefficient leads to a process whose distribution is more "spread out". Consequently, certain functionals of the process, such as "multi-time moment functions" in the case where the state space of the components is a compact interval in $\mathbb{R}_+$, have bigger expectations. This comparison gives a useful tool for studying cluster formation in such interacting systems. At the same time it fills a gap (in an application of the integration by parts formula involving semigroups) in the proof of Proposition 4.10 (jj) of [FG94a] concerning the comparison with an interacting *restricted* Fisher-Wright diffusion. (See also Figure 1 below.)

Using this comparison technique we are able to resolve a problem in the ergodic theory of interacting diffusions in the case where the underlying symmetrized migration term is *recurrent*, and where the state space of a component is *one-sided bounded*. We show that *clustering is universal* in the diffusion coefficient. This had been conjectured in Cox, Greven and Shiga [CGS94a] (see also Shiga [Shi92]). On the way, we obtain a new proof, in the case where the state space of a component is compact, based on the interacting Fisher-Wright diffusion where a well-known duality is available.

Further applications will be contained in the paper [FG94b] on the time-space cluster formation of hierarchically interacting systems in the regime of diffusive clustering, and in [CGS94b] where the relation between finite and infinite systems is studied.

1.1 The model

Consider the following model (compare with [CGS94a]).

Definition 1 (interacting diffusion X) Let $X = \{X_i(t);\, i \in K,\, t \geq 0\}$ be the unique (for each specified initial state $X(0) \in \mathbb{E}$) *strong* solution of the following infinite-dimensional stochastic differential equation

$$\mathrm{d}X_i(t) = \Big(\kappa \sum_j p_{i,j}\big[X_j(t) - X_i(t)\big]\Big)\mathrm{d}t + \sqrt{g\big(X_i(t)\big)}\,\mathrm{d}w_i(t), \qquad i \in K, \tag{1}$$

with values in $\mathbb{E}$.

The ingredients of this equation are as follows:

(a) **(label set)** K denotes a countable non-empty set and is used to label the components of the system.

(b) **(migration parameters)** $p = \{p_{i,j}\,;\, i, j \in K\}$ is a probability transition matrix in K, and κ a non-negative constant. We call p the *migration kernel* and κ the *migration intensity.*

(c) **(driving Brownian motions)** $\{w_i\,;\, i \in K\}$ is a system of independent standard Brownian motions in $\mathbb{R}$ describing the *noise* in the system.

(d) **(diffusion coefficient g)** The *diffusion coefficient* $g : \mathbb{R} \to \mathbb{R}_+$ is assumed to satisfy the following conditions:

  (d1) g is locally Lipschitz continuous,

  (d2) $g = 0$ on the complement of an *open* interval I,

  (d3) g has at most a quadratic growth (if I is unbounded):

$$\limsup_{|r|\to\infty} \frac{g(r)}{r^2} < \infty.$$

(e) (state space $\mathbb{E}$) Let $\gamma = \{\gamma_i\,;\ i \in K\}$ be a (strictly) positive, summable "reference measure" independent of g satisfying

$$\textstyle\sum_i \gamma_i\, p_{i,j} \ \le \Gamma\, \gamma_j\,, \quad j \in K, \qquad \text{for some constant } \Gamma,$$

and put $\mathbb{E} := \{z \in \overline{I}^K;\ \|z\| < \infty\}$ where $\|z\| := \sum_{i\in K} \gamma_i |z_i|$. The convex set $\mathbb{E}$ is endowed with the topology of componentwise convergence.

Write $P_\mu = P^g_\mu$ for the distribution of X if it starts off with the law $\mu = \mathcal{L}(X(0))$, and $P_z = P^g_z$ in the special case $\mu = \delta_z$ (Dirac measure at $z \in \mathbb{E}$). The random initial state $X(0)$ is always assumed to be *independent* of the driving Brownian motions $\{w_i\,;\ i \in K\}$. ◇

Remark 2 (i) Note that the strong solution of (1) is a *Markov process with continuous paths.*
(ii) For the construction of a unique solution for equations of this type, see Shiga and Shimizu [SS80].
(iii) Note also that there is some freedom in the choice of the reference interval I, but additionally also in the choice of the reference measure γ.
(iv) The integrability condition $\|z\| < \infty$ prevents $\sup_{i\in K_d} |z_i|$ from growing too rapidly as $d \to \infty$, where $\{K_d\}$ is any sequence of finite subsets of K which increase to K.
(v) As in [LS81], a reference measure γ can always be defined by

$$\gamma_i \ := \textstyle\sum_{m=0}^{\infty}\ \Gamma^{-m} \sum_j\ \beta_j\, p^{(m)}_{j,i}\,, \qquad i \in K,$$

where $\Gamma > 1$ and $\beta_j > 0$ for all $j \in K$, as well as $\sum_j \beta_j < \infty$. ◇

Remark 3 If a probability law μ on $\overline{I}^K$ satisfies $\sup_{i\in K} \int \mu(\mathrm{d}z)\, |z_i| < \infty$, then $\mu(\mathbb{E}) = 1$ (with $\mathbb{E}$ from (e)). Hence, each such μ may serve as initial law of the Markov process X. ◇

Example 4 (diffusion coefficients) The label set K is often the lattice space $\mathbb{Z}^d$ or a hierarchical group Ξ (see for instance [FG94a] and [Kle95]), whereas for the diffusion coefficient g the following special cases have been intensively studied (see for instance [CG94, CGS94a, Deu94, FG94a, Shi92] and references therein):

		I	$g(r)$ on I
(i)	*Fisher-Wright*	$(0,1)$	$c\,r(1-r)$
(ii)	*Ohta-Kimura*	$(0,1)$	$c\,r^2(1-r)^2$
(iii)	*Feller's branching diffusion*	$(0,+\infty)$	$c\,r$
(iv)	*linear random potential*	$(0,+\infty)$	$c\,r^2$
(v)	*critical Ornstein–Uhlenbeck*	$\mathbb{R}$	c

where c is always a positive constant (scaling factor). ◇

1.2 The comparison result

Before we formulate our comparison result, we introduce the cones inducing the corresponding order relations.

Definition 5 (function cones $\mathbf{F}$ and $\mathbf{F}_0$) Fix a state space $\mathbb{E}$ as introduced in Definition 1 (e). Denote by $\mathbf{F}$ the set of all functions $F : \mathbb{E} \to \mathbb{R}_+$ which depend on finitely many components only, have bounded continuous partial derivatives of orders $m = 0, 1, 2$, and such that the second order partial derivatives $D_i D_j F$ are non-negative, for all $i, j \in K$, where $D_i := \frac{\partial}{\partial z_i}$.

If we additionally require that these functions F are either all non-decreasing, or alternatively all non-increasing, then we write $\mathbf{F}_0$ instead of $\mathbf{F}$. $\diamond$

In particular, such functions F are convex in each single component, but, of course, not necessarily convex on $\mathbb{E}$ (see Example 6 (a) below). Note that the smaller set $\mathbf{F}_0$ is closed with respect to the operation of multiplication (a fact which is used in the multi-time case).

Example 6 (function cone $\mathbf{F}_0$) We mention a typical example for both cases of $\mathbf{F}_0$, a non-decreasing and a non-increasing function F :

(a) **("moment function")** If I is a bounded subinterval of $\mathbb{R}_+$, we fix natural numbers $d \geq 1$, $n_1, ..., n_d \geq 0$, and labels $i_1, ..., i_d \in K$, and set

$$F(z) := z_{i_1}^{n_1} \cdots z_{i_d}^{n_d}, \qquad z \in \mathbb{E}.$$

Note that in general these functions are *not* convex on $\mathbb{E}$.

(b) **("Laplace function")** If I is bounded below, we fix $\lambda_1, ..., \lambda_d \geq 0$ as well as $i_1, ..., i_d \in K$, and put

$$F(z) := \exp\big[-\lambda_1 z_{i_1} - \cdots - \lambda_d z_{i_d}\big], \qquad z \in \mathbb{E}. \qquad \diamond$$

Now we are ready to state our comparison argument concerning the interacting diffusion $X = \{X(t);\ t \geq 0\}$, which for typographical simplification we also write as $\{X_t;\ t \geq 0\}$ (as long as the labeling of components is not needed).

Theorem 1 (comparison argument) *Fix two diffusion coefficients $g_1 \geq g_2$ with a common reference interval I which is bounded above or below, a finite sequence $t_1, ..., t_n \geq 0$ and functions $F_1, ..., F_n \in \mathbf{F}_0$. Then*

$$E_z^{g_1} F_1(X_{t_1}) \cdots F_n(X_{t_n}) \;\geq\; E_z^{g_2} F_1(X_{t_1}) \cdots F_n(X_{t_n}), \qquad z \in \mathbb{E}. \qquad (2)$$

In particular, for all $t \geq 0$ and $F \in \mathbf{F}_0$,

$$E_z^{g_1} F(X_t) \;\geq\; E_z^{g_2} F(X_t), \qquad z \in \mathbb{E}. \qquad (3)$$

The latter conclusion even holds for $F \in \mathbf{F}$.

Remark 7 (extensions of the comparison argument) Theorem 1 can be extended to hold for functions which arise as limits of functions in $\mathbf{F}$ in such a way that the corresponding functionals also converge. $\diamond$

Example 8 (comparison with restricted Fisher-Wright) We mention an intrinsic example for a situation where the comparison theorem is applicable (and which is intensively used in [FG94a], [FG94b], and [Kle95]).

Let g be a diffusion coefficient with reference interval $I = (0,1)$. Assume that g is positive on I, and set

$$g^\varepsilon(r) := c^\varepsilon(r-\varepsilon)^+(1-\varepsilon-r)^+, \quad r \in \mathbb{R}, \qquad I^\varepsilon :\equiv (0,1) = I,$$

where $0 \le \varepsilon < \frac{1}{2}$ and $c^\varepsilon > 0$. Such a g^ε is called a *restricted* (if $\varepsilon > 0$) Fisher-Wright diffusion coefficient related to the interval $(\varepsilon, 1-\varepsilon)$. (Figure 1.) The

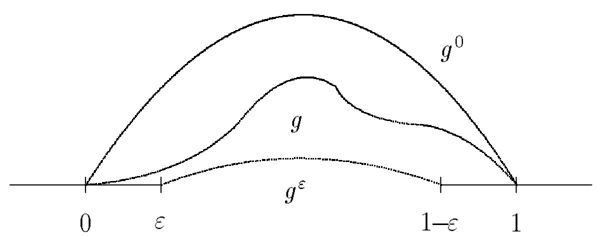

Figure 1: (restricted) Fisher-Wright bounds for g with support $(0,1)$

interacting diffusion with this "reference diffusion coefficient" g^ε and initial state X_0 in $[\varepsilon, 1-\varepsilon]^K$ can be studied using delayed coalescing random walks which are dual to interacting Fisher-Wright diffusions (Shiga [Shi80]). In the sense of the physics literature this is therefore an explicitly solvable model. (A similar explicitly solvable case is the interacting Feller's branching diffusion of Example 4 (iii), which can also be used in comparison arguments.) Note that by our assumptions on g, for each $\varepsilon > 0$ sufficiently small one can always find constants $c^0, c^\varepsilon > 0$ such that $g^0 \ge g \ge g^\varepsilon$. Using "moment functions" F as in Example 6 (a), Theorem 1 provides bounds of all higher and "mixed" moments of X with respect to P_z^g by the corresponding ones in the case of interacting (restricted) Fisher-Wright diffusions. This comparison is useful for the following reasons. First of all, statements on interacting diffusions are frequently proved by the method of moments. Second, limiting statements on the cluster formation as in Theorems 1-5 of [FG94a], in the special case of (restricted) Fisher-Wright diffusion coefficients, do *not* depend on the scaling factor c^ε (and are continuous in ε). Therefore the comparison theorem is a powerful tool for extending results from the Fisher-Wright case to general diffusion coefficients g with support $(0,1)$ (*universality*). $\diamond$

1.3 Applications

Our *main application* of the comparison theorem in this paper is a result on the long-time behavior of interacting diffusions in the *recurrent* case, which covers new classes of systems, and even simplifies proofs for some known cases, as for example, interacting Feller's branching diffusions (super-random walks). In addition, we shall sketch some applications of the comparison theorem in the *transient* case.

To arrive at a simple form for the next theorem we require additional properties of the model.

Assumption 9 (recurrence) In Definition 1 we also assume:

(a) K is a (countable) *Abelian group.*

(b) The migration kernel p is irreducible, *homogeneous* ($p_{i,j} = p_{0,j-i}$), and the symmetrized kernel $\widehat{p}_{i,j} := \frac{1}{2}(p_{i,j} + p_{j,i})$ is *recurrent.*

(c) The diffusion coefficient g is *positive* on the (bounded or unbounded) reference interval $I =: (a,b)$. $\diamond$

Note that by the assumed homogeneity of p, a reference measure γ, needed for the construction of a state space $\mathbb{E}$ as required in Definition 1 (e), exists (recall Remark 2 (v)).

Assumption 10 (homogeneity) The initial law $\mu = \mathcal{L}\big(X(0)\big)$ is assumed to be any *homogeneous* law on $\overline{I}^K$ (that is invariant with respect to the spatial shift induced by the group action), satisfying $E_\mu\big|X_0(0)\big| < \infty$. Set $\theta := E_\mu X_0(0)$. $\diamond$

Note that this assumption makes sense, since such laws μ are supported by $\mathbb{E}$ (recall Remark 3).

Remark 11 If one wants to drop condition (a) in Assumption 9, analogs of (4) below can still be shown if instead of (b) one works with a pair of independent Markov chains which meet infinitely often almost surely (cf. Shiga [Shi80]). $\diamond$

The result we now want to state says that for I bounded above or below, under Assumptions 9, 10, the interacting diffusion *clusters* for *all* diffusion coefficients g (*universality*). Clustering means that for large times, locally, all components almost agree. In fact, with Theorem 1 and the ergodic theorem in the interacting Fisher-Wright case alone, which is easily handled via duality, we are able to derive the following result. Here $\underline{a}$ denotes the constant state $\underline{a}_i \equiv a$.

Theorem 2 (clustering) *For I bounded above or below, under Assumptions 9 and 10,*

$$\mathcal{L}(X_t) \underset{t\to\infty}{\Longrightarrow} \begin{cases} \dfrac{b-\theta}{b-a}\,\delta_{\underline{a}} + \dfrac{\theta-a}{b-a}\,\delta_{\underline{b}}, & \text{if} \quad a,b \in \mathbb{R}, \quad \kappa > 0, \\ \delta_{\underline{a}}, & \text{if} \quad a \in \mathbb{R}, \quad b = +\infty. \end{cases} \tag{4}$$

It remains an *open problem* to prove that in the remaining case $-a = b = +\infty$

$$\mathcal{L}(X_t) \underset{t\to\infty}{\Longrightarrow} \tfrac{1}{2}\big(\delta_{-\underline{\infty}} + \delta_{+\underline{\infty}}\big)$$

(in a suitable sense), which is only known for the interacting critical Ornstein-Uhlenbeck diffusion $g(r) \equiv c > 0$ on $\mathbb{R}$, that is explicitly solvable using Gaussian techniques.

Remark 12 The universality for *finite* a and b was known before, see Cox and Greven [CG94]. But for g with *unbounded* support, only special cases have been handled so far. In fact, extinction behavior for interacting diffusions with linear potential (Example 4 (iv)) had been studied in Shiga [Shi92]; extinction properties of spatial branching models related to the interacting Feller's branching diffusion (super-random walk) of Example 4 (iii) are also well-known; cf. e.g. Dawson [Daw77]. ◇

Remark 13 In the case $I = (0, \infty)$, treating initial states with $E_\mu\big|X_0(0)\big| = \infty$ is a bit more subtle, since the limit point δ_∞ may appear; cf. Bramson et al. [BCG94], Dawson et al. [DFFP86]. However, using the relatively well-understood interacting Feller's branching diffusion (super-random walk), it is possible to use Theorem 1 to get here results in the class of processes where $g(r)/r \to 0$ or ∞ as $r \to \infty$ as well. ◇

Further applications When the symmetrized migration kernel is *transient,* the long-term behavior of X is relatively well understood (see Cox and Greven [CG94], Deuschel [Deu94], and Shiga [Shi92]). Nevertheless, even in this situation, the comparison theorem yields new information.

For instance, consider the case $I = (0, +\infty)$, $g(r) = c\,r^2$ of Example 4 (iv) with $K = \mathbb{Z}^d$ and an irreducible homogeneous migration kernel p of finite range with transient symmetrized kernel $\widehat{p}$, and $\kappa > 0$. It follows from Shiga's Theorems 1.1 and 1.2 in [Shi92], that if $X_i(0) \equiv \theta > 0$, then there are (strictly) positive constants $c_1 < c_2$ such that

$$\mathcal{L}(X_t) \underset{t\to\infty}{\Longrightarrow} \begin{cases} \nu_\theta\,, & \text{if} \quad c < c_1\,, \\ \delta_0\,, & \text{if} \quad c > c_2\,, \end{cases} \tag{5}$$

where ν_θ is a non-trivial invariant measure with "full" intensity θ. In particular, local extinction holds if $c > c_2$ but not if $c < c_1$. (Even more, the components $X_i(t)$ tend to zero exponentially fast in the case $c > c_2$.) It follows immediately from the comparison Theorem 1 that there exists a *critical* c^* such that local extinction holds for all $c > c^*$, but not for $c < c^*$, and Shiga's result (5) implies that $c^* \in (0, +\infty)$.

Another application of the comparison theorem is the following. Suppose $K = \mathbb{Z}^d$, $\kappa > 0$, the migration kernel p is irreducible and homogeneous, $\widehat{p}$

is transient, and g is linked with p by the following: The limit superior in Definition 1 (d3) is (strictly) smaller than $1/\widehat{q}_{0,0}$ where $\widehat{q}$ is the Green's function of the continuous time random walk in K with jump rates given by $\kappa\widehat{p}$ (see condition (1.2) in [CGS94a]). Then to each $\theta \in I$ there is an invariant measure $\nu_\theta = \nu_\theta^g$ with density θ (see [CG94] and [Shi92]). In the finite systems scheme of [CGS94a] there is a certain non-linear transformation $g \mapsto g^*$ given by $g^*(\theta) = E^g_{\nu_\theta}\, g\big(X_0(0)\big)$. Using Lemma 2.11 (a) and Proposition 2.3 (c) of [CGS94a], and the fact that ν_θ is invariant, we have

$$E^g_{\nu_\theta}\, X_0^2(t) = E^g_{\nu_\theta}\, X_0^2(0) = \theta^2 + \widehat{q}_{0,0}\, E^g_{\nu_\theta}\, g\big(X_0(0)\big) = \theta^2 + \widehat{q}_{0,0}\, g^*(\theta).$$

A suitable approximation procedure can be employed to show that the conclusion of the comparison theorem holds here with the choice $F(z) = z_0^2$. It then follows easily that

$$g_1 \le g_2 \quad \text{implies} \quad g_1^* \le g_2^*. \tag{6}$$

This monotonicity property is very useful in studying the nonlinear map $g \longrightarrow g^*$ (see [BCGdH95]).

The rest of the paper is organized as follows. We prove Theorem 1 in the next section, and Theorem 2 in Section 3.

2 Proof of the comparison theorem

The key idea of the proof of Theorem 1 is to use an integration by parts formula for semigroups combined with a preservation property of the function cones $\mathbf{F}_0$ and $\mathbf{F}$ under the interacting diffusion semigroup. The proof will first treat a simple case, and then later generalize.

Assumption 14 (simplifications) Let $K = \{1, ..., d\}$, $I = (a, b)$ with finite $a \le b$, and assume that g is twice continuously differentiable on $\mathbb{R}$.

Consequently, we start with a nice finite-dimensional situation. Under these simplifications, we first recall that our semigroups preserve smooth functions in § 2.1. Then preservation of $\mathbf{F}$ in the case of pure diffusion ($\kappa = 0$) is handled in § 2.2. The case of pure migration is handled in § 2.3, where we also put these two cases together using Trotter's product formula. Furthermore, for functions in the smaller set $\mathbf{F}_0$ we extend to the multi-time point case. Then, under Assumption 14 for both g_1 and g_2 with a common I, the comparison theorem is verified in § 2.4. Finally, the restrictions of Assumption 14 will be removed in § 2.6.

2.1 Preservation of smoothness

Impose Assumption 14. Then $\mathsf{E} = \overline{I}^d$, which can be considered as a compact topological subspace of $\mathbb{R}^d$, that is the convergence in E can be described by the Euclidean norm $|\cdot|$.

Let $\mathbf{C} = \mathbf{C}(\mathsf{E})$ denote the Banach space of all (bounded) continuous functions $h : \mathsf{E} \to \mathbb{R}$ with the supremum norm of uniform convergence, and $\mathbf{C}^2 = \mathbf{C}^2(\mathsf{E})$ the subset of all those functions in $\mathbf{C}$ which have (bounded) continuous partial derivatives on E of orders $0, 1, 2$.

Denote by $\mathsf{S} = \mathsf{S}^g$ the strongly continuous contraction *semigroup* associated with the Markov process X of Definition 1,

$$\mathsf{S}_t^g h\,(z) = E_z^g h(X_t), \qquad z \in \mathsf{E} = \overline{I}^d,$$

acting on $\mathbf{C}$. This semigroup has as its *generator* $\mathsf{G} = \mathsf{G}^g$ the closure of the following operator (also denoted by G) acting on $\mathbf{C}^2$:

$$\mathsf{G}^g \;:=\; \kappa \sum_{i,j} \left(p_{i,j} - \delta_{i,j}\right) z_j D_i + \tfrac{1}{2} \sum_i g(z_i) D_i^2, \qquad z \in \mathsf{E} = \overline{I}^d. \tag{7}$$

For convenience, we expose the following fact (see Theorem 8.4.3 in [GS69]) as a lemma.

Lemma 15 (preservation of smoothness) *Suppose Assumption* 14 *holds. Then* $\mathbf{C}^2$ *is preserved under the semigroup* S, *that is* $\mathsf{S}_t f \in \mathbf{C}^2$, $f \in \mathbf{C}^2$, $t \geq 0$.

2.2 Preservation of F under the drift-less diffusion

Here we prove the following result.

Proposition 16 (preservation of F under the pure diffusion) *Impose Assumption* 14 *and* $\kappa = 0$. *If* $F \in \mathbf{F}$ *(or* $\mathbf{F}_0$*) then for each fixed* $t > 0$, *the function* $z \mapsto \mathsf{S}_t F(z)$ *on* E *also belongs to* $\mathbf{F}$ *(or* $\mathbf{F}_0$, *respectively).*

Proof Fix $F \in \mathbf{F}$ and $t > 0$. Obviously, $\mathsf{S}_t F$ is again non-negative and has the required smoothness by Lemma 15.

Step 1° For $i \in K$, denote by e_i the ith unit vector in $\mathbb{R}^d$. Fix $i, j \in K$ for the moment (not necessarily different). Let us say that $\mathbf{u} = (u^0, u^1, u^2, u^{12}) \in I^{4d}$ forms an *ij–rectangle* (in I^d) if

$$\begin{aligned} u^0 &= z \\ u^1 &= z + h_1 e_i \\ u^2 &= z + h_2 e_j \\ u^{12} &= z + h_1 e_i + h_2 e_j \end{aligned}$$

for some $z \in I^d$ and positive h_1 and h_2.

Let f be a twice continuously differentiable function on I^d. A little calculus shows that $D_i D_j f \geq 0$ on I^d for all $i, j \in K$, *if and only if*

$$f(u^{12}) - f(u^2) - f(u^1) + f(u^0) \geq 0 \tag{8}$$

whenever $\mathbf{u}$ is an ij–rectangle (in I^d), $i, j \in K$.

Step 2° In order to show that the cone $\mathbf{F}$ is preserved, it remains to show that $D_i D_j \mathsf{S}_t F \geq 0$ on I^d for all $i, j \in K$. By the previous step, our task is to show that if we indeed fix $i, j \in K$ and an ij–rectangle $\mathbf{u} = (u^0, u^1, u^2, u^{12}) = \mathbf{u}(z, h_1, h_2)$ in I^d, then

$$\mathsf{S}_t F(u^{12}) - \mathsf{S}_t F(u^2) - \mathsf{S}_t F(u^1) + \mathsf{S}_t F(u^0) \geq 0. \tag{9}$$

We will do this by constructing coupled versions X^0, X^1, X^2, X^{12} of the diffusion X which start at the u^0, u^1, u^2, u^{12}, respectively. For this purpose, we will modify the components X_i and X_j of our d–dimensional diffusion X (with independent component diffusions X_k, $k \in K$) starting at $X(0) = z$ in several ways.

Let Y^1, Y^2, Y^{12} denote one-dimensional (drift-less) diffusions each with diffusion coefficient g, starting at $z_i + h_1 e_i$, $z_j + h_2 e_j$, $z_i + h_1 e_i + h_2 e_j$, respectively. We assume that (Y^1, Y^2) is *coupled* with (X_i, X_j) such that $X_i \leq Y^1$ and $X_j \leq Y^2$. This coupling can easily be realized by working with the *same* pair (w_i, w_j) of driving Brownian motions (cf. [RY91, Theorem 9.3.7]).

Set $X^0 = X$. Set $X^1 = X$, but replace the i^{th} component X_i^1 with Y^1, choosing Y^1 independent of $\{X_k ;\ k \neq i\}$. Set $X^2 = X$, but replace the j^{th} component X_j^2 with Y^2, choosing Y^2 independent of $\{X_k ;\ k \neq j\}$. For the case $i \neq j$, set $X^{12} = X$, but replace (X_i^{12}, X_j^{12}) with (Y^1, Y^2), choosing Y^1 and Y^2 so that the components of X^{12} are independent. For the case $i = j$, choose Y^{12} independent of $\{X_k ;\ k \neq i\}$ such that $Y^{12} \geq Y_1, Y_2$, and set $X^{12} = X$, but replace X_i^{12} with Y^{12}. By the couplings, with probability one, regardless of whether i and j are different or not, the points $X^0(t), X^1(t), X^2(t), X^{12}(t)$ form an ij–rectangle. Hence, by step 1°

$$F(X_t^{12}) - F(X_t^2) - F(X_t^1) + F(X_t^0) \geq 0.$$

Taking expectations, we obtain (9) as required, finishing the proof in the case of $\mathbf{F}$.

Step 3° If $F \in \mathbf{F}_0$, the preservation of monotonicity can easily be seen by a coupling argument as in step 2°, since all components evolve independently. ■

2.3 Preservation under the simplifications

Here we generalize Proposition 16 to include the migration case $\kappa > 0$ and also multiple time points. As announced, the key idea is here to use Trotter's product formula for the semigroups arising by considering $\kappa = 0$ and $g = 0$, respectively.

Proposition 17 (preservation under simplifications) *Under Assumption 14, for each finite sequence $F_1, ..., F_n \in \mathbf{F}_0$ and time points $t_1, ..., t_n \geq 0$, the function*

$$z \mapsto E_z^g F_1(X_{t_1}) \cdots F_n(X_{t_n}), \qquad z \in \mathsf{E},$$

belongs to $\mathbf{F}_0$. In the case $n = 1$, the statement is correct even for $\mathbf{F}_0$ replaced by $\mathbf{F}$.

Proof Preservation of non-negativity is again trivial. The proof of the remaining statements is by induction on the number n of time points.

$1°$ (*first step of induction*) For $n = 1$ drop the index 1 in notation, that is, look at

$$H_t(z) := E_z^g F(X_t), \qquad z \in \mathsf{E} = \overline{I}^d, \tag{10}$$

for a fixed $t > 0$ and $F \in \mathbf{F}$ (respectively $\mathbf{F}_0$). Assume that $a = b$ in Assumption 14, i.e., that $g(r) \equiv 0$. Then X degenerates to a *deterministic* process. In this case we can explicitly solve the linear system (1):

$$X_i(t) = \sum_{j \in K} p(t, i, j)\, z_j\,, \qquad i \in K = \{1, ..., d\}. \tag{11}$$

Here z is the initial state $X(0)$, and $p(t, i, j)$ are the transition probabilities of the continuous-time Markov chain in K with jump rates $\kappa\, p_{i,j}$. Hence, in this pure migration case, H_t can be written as

$$H_t(z) = F\bigl(X(t)\bigr) \quad \text{with } X(t) \text{ from (11).} \tag{12}$$

The preservation of smoothness was already clear from Lemma 15. By the chain rule,

$$D_i D_j H_t(z) = \sum_{i', j'} D_{i'} D_{j'} F\bigl(X_t\bigr) p(t, i', i)\, p(t, j', j) \tag{13}$$

which must be non-negative since $D_i D_j F \geq 0$ everywhere. Consequently, $H_t \in \mathbf{F}$, and the analogous statement is true for $\mathbf{F}_0$ instead of $\mathbf{F}$.

The case $\kappa = 0$ is handled by Proposition 16. For the general case, we decompose the interval $[0, t]$ into small pieces of length t/k and apply alternately the diffusions with $\kappa = 0$ and $g = 0$, with t replaced by t/k. More specifically, consider

$$H_{k,t}(z) := \bigl[\mathsf{S}_{t/k}^{(2)} \mathsf{S}_{t/k}^{(1)}\bigr]^k F\,(z), \qquad z \in \mathsf{E} = \overline{I}^d, \quad k \geq 1, \tag{14}$$

where $\mathsf{S}^{(1)}$ refers to the semigroup of independent diffusions ($\kappa = 0$), and $\mathsf{S}^{(2)}$ to the degenerate semigroup related to the deterministic process ($g = 0$). Since $F \in \mathbf{F}$, each successive step results in a function in $\mathbf{F}$. We end up in $\mathbf{F}$ with the whole chain of $2k$ operations in (14). That is, $H_{k,t} \in \mathbf{F}$ for each k. By *Trotter's product formula* (see for instance Corollary 1.6.7 of [EK86], working with $\mathbf{C}^2$ as a core for the generator G), we get the limit $\lim_{k \to \infty} H_{k,t} = \mathsf{S}_t F = H_t$ in $\mathbf{C}$. By Lemma 15, $H_t \in \mathbf{C}^2$, and hence by step $1°$ of Proposition 16, it suffices to show that H_t satisfies (8) for all i, j–rectangles. But since this is true for each $H_{k,t}$, the non-negativity is maintained also in the limit as $k \to \infty$. Consequently, H_t belongs to $\mathbf{F}$. This finishes the proof in the case $n = 1$ for $\mathbf{F}$. If $F \in \mathbf{F}_0$, then again Trotter yields the monotonicity claim, giving $H_t \in \mathbf{F}_0$. This finishes the first step of induction.

2° (*induction step*) Now assume that $n > 1$. Since $\mathbf{F}_0$ is closed with respect to multiplications, without loss of generality we can assume that $0 < t_1 < \cdots < t_n$. Then by the Markov property the expression under consideration can be written as the following product of two functions, one with a single time point and one with $n-1$ time points:

$$E_z^g F_1(X_{t_1}) E_{X_{t_1}}^g F_2(X_{t_2 - t_1}) \cdots F_n(X_{t_n - t_1}).$$

Because $\mathbf{F}_0$ is closed under multiplication, the proof can easily be completed by induction. ■

2.4 Proof of the comparison theorem under simplifications

Fix g_1 and g_2 satisfying Assumption 14 with a common I, and $g_1 \geq g_2$. Again without loss of generality we may assume that $0 < t_1 < \cdots < t_n$.

The proof of (2) in this simplified case is by induction over n, the number of time points considered. Start with $n = 1$, and suppose only that F_1 belongs to $\mathbf{F}$. We have to show that

$$\mathsf{S}_{t_1}^{g_1} F_1(z) = E_z^{g_1} F_1(X_{t_1}) \geq E_z^{g_2} F_1(X_{t_1}) = \mathsf{S}_{t_1}^{g_2} F_1(z), \qquad z \in \mathbb{E} = \overline{I}^d. \tag{15}$$

By continuity, we may restrict to $z \in I^d$. By the *integration by parts formula*

$$\mathsf{S}_{t_1}^{g_1} - \mathsf{S}_{t_1}^{g_2} = \int_0^{t_1} \mathrm{d}s \; \mathsf{S}_{t_1 - s}^{g_2} (\mathsf{G}^{g_1} - \mathsf{G}^{g_2}) \mathsf{S}_s^{g_1} \tag{16}$$

(see for instance p. 367 in [Lig85]), it suffices to demonstrate that

$$(\mathsf{G}^{g_1} - \mathsf{G}^{g_2}) \mathsf{S}_s^{g_1} F_1 \geq 0, \qquad 0 \leq s \leq t_1, \tag{17}$$

on $\mathbb{E}$. Note that $\mathsf{S}_s^{g_1} F_1$ belongs to $\mathbf{C}^2$ by Lemma 15, hence is also in the domain of G^{g_2}. By the form of the generators (recall (7)),

$$\mathsf{G}^{g_1} - \mathsf{G}^{g_2} = \tfrac{1}{2} \textstyle\sum_i \left[g_1(x_i) - g_2(x_i) \right] D_i^2.$$

Since $g_1 \geq g_2$ by assumption, for the proof of (17) it therefore suffices to show that for fixed s

$$\mathsf{S}_s^{g_1} F_1(z) \quad \text{is } \textit{convex} \text{ in each component } z_i, \;\; i \in K, \;\text{ of }\; z \in \mathbb{E}. \tag{18}$$

But this follows from the preservation Proposition 17. Consequently, the inequality (15), hence (2) is true in the case $n = 1$.

Consider $n \geq 2$ and assume that the $F_1, ..., F_n$ belong to $\mathbf{F}_0$. For a preparation of the induction step, we first rewrite the inequality (2) in a more convenient form, using the fact that $\mathbf{F}_0$ is closed under multiplication. Namely, using the Markov property at time t_1 and time-homogeneity of X, we see that (2) becomes

$$\mathsf{S}_{t_1}^{g_1} F^1(z) = E_z^{g_1} F^1(X_{t_1}) \geq E_z^{g_2} F^2(X_{t_1}) = \mathsf{S}_{t_1}^{g_2} F^2(z), \qquad z \in \mathbb{E}, \tag{19}$$

by setting

$$F^m(x) := F_1(x) E_x^{g_m} F_2(X_{t_2 - t_1}) \cdots F_n(X_{t_n - t_1}), \qquad x \in \mathbb{E}, \quad m = 1, 2. \tag{20}$$

Assume now that (2), respectively (19), is valid for some $n-1 \geq 1$ (*induction hypothesis*). Then, by (2) and the non-negativity of F_1, from the definition (20) we immediately get $F^1 \geq F^2$ on $\mathbb{E}$. Then the relation (19) for $n \geq 2$, hence (2) for $n \geq 2$, will follow from the non-negativity of the semigroups $\mathsf{S}^{g_1}, \mathsf{S}^{g_2}$ once we prove (15) with F_1 replaced by F^1. As in the case $n = 1$, for this we need to know (18), with F_1 replaced by F^1. By the definition (20) of F^1, we may return to the original expression:

$$\mathsf{S}_s^{g_1} F^1(z) = E_z^{g_1} F_1(X_s) F_2(X_{t_2}) \cdots F_n(X_{t_n}). \tag{21}$$

Again by the preservation Proposition 17, the needed componentwise convexity property holds. This finishes the induction step, hence the proof of the comparison theorem under Assumption 14. ■

2.5 Two approximation procedures

To complete the proof of Theorem 1, we have to remove the Assumption 14. This will essentially be based on the following two approximation procedures. To this purpose, fix $n \geq 1$, $t_1, ..., t_n \geq 0$, as well as $F_1, ..., F_n \in \mathbf{F}_0$, respectively $F_1 \in \mathbf{F}$ if $n = 1$.

Lemma 18 (approximation by finite K) *Let I be bounded and g be twice continuously differentiable on $\mathbb{R}$. Consider finite sets $K_1 \subseteq K_2 \subseteq \cdots \uparrow K$. Let X^ℓ denote the process obtained from X according to Definition* (1) *by the following modification. For $i \notin K_\ell$, freeze X_i (that is put $X_i(t) \equiv X_i(0)$), whereas for $i \in K_\ell$, restrict the summation in* (1) *to $j \in K_\ell$. Then,*

$$E_z^g F_1(X_{t_1}^\ell) \cdots F_n(X_{t_n}^\ell) \underset{\ell \to \infty}{\longrightarrow} E_z^g F_1(X_{t_1}) \cdots F_n(X_{t_n}).$$

Proof See the proof of Theorem 2.1 in [SS80]. ■

Lemma 19 (continuity in g) *Let $X^0, X^1, ...$ denote interacting diffusion according to Definition 1 which might differ only in their diffusion coefficients*

$g_0, g_1, \dots,$ respectively. Assume that the $g_0, g_1, \dots,$ have a common reference interval I, that $g_\ell \to g_0$ pointwise as $\ell \to \infty$, and that

$$g_\ell(r) \le c_1 + c_2 r^2, \quad r \in \mathbb{R},\ \ell \ge 1, \quad \textit{for some constants } c_1, c_2. \tag{22}$$

Then

$$E_z^{g_\ell} F_1(X_{t_1}) \cdots F_n(X_{t_n}) \underset{\ell\to\infty}{\longrightarrow} E_z^{g_0} F_1(X_{t_1}) \cdots F_n(X_{t_n}).$$

Sketch of proof Take $T \ge t_1, \dots, t_n$. For fixed $z \in \mathbb{E}$, the family of laws $\{P_z^{g_\ell};\ \ell \ge 1\}$ on $\mathbf{C}([0,T], \mathbb{E})$ with the topology of uniform convergence is tight. Moreover, any of its subsequential limits as $\ell \to \infty$ must satisfy the martingale problem related to the interacting diffusion X^0. But there is a unique solution to that problem, namely $\{P_z^{g_0};\ z \in \mathbb{E}\}$. Hence $P_z^{g_\ell}$ converges weakly to $P_z^{g_0}$, for each z, and the claim follows. ∎

2.6 Completion of the proof of the comparison theorem

Fix diffusion coefficients $g_1 \ge g_2$ with a common reference interval I, and $t_1, \dots, t_n \ge 0$ as well as $F_1, \dots, F_n \in \mathbf{F}_0$, respectively $F_1 \in \mathbf{F}$ if $n = 1$.

Step 1° To complete the proof of Theorem 1, assume first additionally that I is bounded and that the g_1, g_2 are twice continuously differentiable on $\mathbb{R}$. Approximate K by finite sets K^ℓ as in Lemma 18. Then for the corresponding processes X^ℓ the claims (2) and (3) had been proved already in § 2.4. By Lemma 18, they then also hold for the limiting process X.

Step 2° If I is not bounded, for each $\ell \ge 1$ choose a twice continuously differentiable function $\varphi_\ell : \mathbb{R} \mapsto [0,1]$ with compact closed support such that $\varphi_\ell \uparrow 1$ pointwise as $\ell \to \infty$. Then the claims hold for the g_1, g_2 replaced by $\varphi_\ell g_1, \varphi_\ell g_2$, respectively, by the previous step of proof. But by Lemma 19 we may pass to the limit as $\ell \to \infty$ yielding the claims also for an unbounded I.

Step 3° If finally the g_1, g_2 are not smooth, approximate them pointwise by twice continuously differentiable function g_1^ℓ, g_2^ℓ, $\ell \ge 1$, in such a way that (22) holds. (To realize this, take a twice continuously differentiable function $h : \mathbb{R} \mapsto \mathbb{R}_+$ with support $(0,1)$, and with integral 1. Set

$$g_m^\ell(r) := \ell \int \mathrm{d}r'\ h\big(\ell\,(r' - r)\big)\, g_m(r'), \qquad r \in \mathbb{R}, \quad \ell \ge 1, \quad m = 1, 2.$$

For ℓ fixed and r varying in a bounded set, the domains of integration can be chosen to be uniformly bounded. Hence, differentiating with respect to r shows that these functions have the required smoothness. On the other hand, from the identity $g_m^\ell(r) = \ell \int_0^1 \mathrm{d}r'\ h(\ell r')\, g_m(r' + r)$ we easily get the domination (22).)

Finish the proof by Lemma 19. ∎

3 Proof of the clustering theorem

Because of [CG94], we may restrict our attention to the second convergence claim in (4), even though we shall outline a new proof of the first statement in step 2° below.

The idea of proof of Theorem 2 is to bound the diffusion coefficient g below by appropriate Fisher-Wright diffusion coefficients on large intervals, and exploit the comparison argument for suitable functionals.

1° (*Proof of the second convergence statement*) Without loss of generality, we may put $a = 0$, that is $I = (0, +\infty)$. Take an $\varepsilon_0 \in (0,1)$. First consider an initial distribution μ which besides Assumption 10 additionally satisfies

$$\mu\Big(\varepsilon_0 \leq z_i \leq \varepsilon_0^{-1},\ \ i \in K\Big) = 1. \tag{23}$$

It then suffices to show that for the Laplace functional of X_t

$$\liminf_{t\to\infty} E_\mu^g \exp\big[-\langle\lambda, X_t\rangle\big] \geq 1$$

for each $\lambda \in \mathbb{R}_+^K$ with $\lambda_i \neq 0$ for only finitely many i. In other words, we may consider a "Laplace function" $F \in \mathbf{F}_0$ as written in Example 6 (b), and we have to estimate $E_\mu^g F(X_t)$ from below appropriately.

For each $\varepsilon \in (0, \varepsilon_0)$ sufficiently small, we find a constant $c^\varepsilon > 0$ such that $g \geq g^\varepsilon$ with g^ε defined by

$$g^\varepsilon(r) := c^\varepsilon (r - \varepsilon)^+ (\varepsilon^{-1} - r)^+, \qquad r \in \mathbb{R},$$

(compare with the lower estimate in Figure 1). By the comparison Theorem 1, for F as given in Example 6 (b), we get

$$E_\mu^g F(X_t) \ \geq\ E_\mu^{g^\varepsilon} F(X_t). \tag{24}$$

Assuming for the moment that $\kappa > 0$, then by the first convergence statement in (4), taking into consideration (23), X_t with respect to $P_\mu^{g^\varepsilon}$ has a clustered limit in law as $t \to \infty$ denoted by X_∞. Applying this to the continuous bounded function F of Example 6 (b) (recall that $z \geq 0$ by the assumption $a = 0$) yields

$$E_\mu^{g^\varepsilon} F(X_t) \ \underset{t\to\infty}{\longrightarrow}\ EF(X_\infty) \ \geq\ \frac{\varepsilon^{-1} - \theta}{\varepsilon^{-1} - \varepsilon} \exp\big[-(\lambda_1 + \cdots + \lambda_n)\,\varepsilon\big].$$

But the latter term converges to 1 as $\varepsilon \downarrow 0$.

Consequently, $\mathcal{L}(X_t) \Longrightarrow \delta_{\underline{0}}$ as $t \to \infty$ which proves the second part of the claim (4) in the case $\kappa > 0$ and of a μ satisfying the restriction (23).

If $\kappa = 0$, then all components are independent, and we can apply the previous conclusions separately to each component. Thus for each component we get the limit δ_0, which also combines to $\delta_{\underline{0}}$.

In order to treat a general initial distribution μ, for $\varepsilon \in (0,1)$ let Γ^ε denote the image law on $\mathbb{E} \times \mathbb{E}$ of μ under the mapping

$$z_i \mapsto \left[z_i\,,\, \varepsilon \vee z_i \wedge \varepsilon^{-1}\right], \qquad i \in K.$$

Note that the first marginal law of Γ^ε is μ, whereas the second, *truncated* one, again satisfies Assumption 10. Now it is easy to show that if we construct a bivariate process $[X, X^\varepsilon]$ starting with law Γ^ε and such that X and X^ε satisfies (1) but using the *same* driving Wiener processes for both (*coupling*), then

$$E^g_{\Gamma^\varepsilon}\left|X_i(t) - X^\varepsilon_i(t)\right| \le E^g_{\Gamma^\varepsilon}\left|X_0(0) - X^\varepsilon_0(0)\right|, \qquad i \in K, \quad t \ge 0,$$

(see [FG94a, Proof of Lemma 4.6]). But the r.h.s. converges to 0 as $\varepsilon \downarrow 0$. Hence, the claim holds for general μ.

2° (*outline of a Proof of the first convergence statement*) Without loss of generality, we may put $a = 0$ and $b = 1$. Since $E^g_\mu X_i(t) \equiv \theta$, it suffices to show that, under $\kappa > 0$, for $0 < \varepsilon < \frac{1}{2}$, and all $i, j \in K$,

$$P^g_\mu\Big(X_i(t) \in [\varepsilon,\, 1-\varepsilon]\Big) \;+\; P^g_\mu\Big(\left|X_i(t) - X_j(t)\right| \ge \varepsilon\Big) \xrightarrow[t\to\infty]{} 0.$$

This result is known for interacting Fisher-Wright diffusions using duality (Shiga [Shi80]). Now proceed as in step 1°, but with replacing "Laplace functions" in $\mathbf{F}_0$ by "second moment functions" $F(z) := z_i z_j - z_i$ in $\mathbf{F}$. We leave the details to the reader (cf. [CG94]). ■

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